

# The first order correction to the exit distribution for some random walks

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## Abstract

We study three different random walk models on several two-dimensional lattices by Monte Carlo simulations. One is the usual nearest neighbor random walk. Another is the nearest neighbor random walk which is not allowed to backtrack. The final model is the smart kinetic walk. For all three of these models the distribution of the point where the walk exits a simply connected domain  $D$  in the plane converges weakly to harmonic measure on  $\partial D$  as the lattice spacing  $\delta \rightarrow 0$ . Let  $\omega(0, \cdot; D)$  be harmonic measure for  $D$ , and let  $\omega_\delta(0, \cdot; D)$  be the discrete harmonic measure for one of the random walk models. Our definition of the random walk models is unusual in that we average over the orientation of the lattice with respect to the domain. We are interested in the limit of  $(\omega_\delta(0, \cdot; D) - \omega(0, \cdot; D))/\delta$ . Our Monte Carlo simulations of the three models lead to the conjecture that this limit equals  $c_{M,L} \rho_D(z)$  times Lebesgue measure with respect to arc length along the boundary, where the function  $\rho_D(z)$  depends on the domain, but not on the model or lattice, and the constant  $c_{M,L}$  depends on the model and on the lattice, but not on the domain. So there is a form of universality for this first order correction. We also give an explicit formula for the conjectured density  $\rho_D$ .

# 1 Introduction

Let  $D$  be a simply connected domain in the plane which contains the origin. We introduce a lattice with spacing  $\delta$  and consider a nearest neighbor random walk that starts at the origin. We are interested in the point where the walk exits the domain. In the limit that the lattice spacing  $\delta$  goes to zero, the distribution of this random point converges to harmonic measure which is the distribution of the point where a Brownian motion exits the domain. In this paper we are interested in the leading order term (in  $\delta$ ) in the difference between the random walk exit distribution and harmonic measure. Our simulations will indicate that this leading order term is proportional to  $\delta$ . So we will refer to it as the first order correction.

In addition to the nearest neighbor random walk, which we will refer to as the ordinary random walk, we study two other random walks for which the scaling limit of the exit distribution is also harmonic measure. One model is a nearest neighbor random walk that is not allowed to backtrack. In other words, at each step the walk picks with equal probability one of its nearest neighbors other than the site it just came from. The third model is the smart kinetic walk, also known as the infinitely growing self-avoiding walk [9, 6]. At each time step the walk is only allowed to jump to a nearest neighbor that has not been visited before and is not a trapping site at that time. A site is a trapping site at time  $n$  if there is no nearest neighbor path from the site to the exterior of the domain through sites that are unoccupied at time  $n$ . See [5] and references therein for more details.

The convergence of the exit distribution of the ordinary random walk to harmonic measure is a well known result. An exposition may be found in [7]. A proof of the functional central limit theorem for the random walk without backtracking on the square lattice may be found in [2]. The smart kinetic walk on the hexagonal lattice is equivalent to the exploration process for critical percolation on the triangular lattice. In this case it has been proved that the exit distribution converges to harmonic measure [1, 3, 8, 10]. For other lattices this is only a conjecture, but with good numerical support [5]. (The definition of the exploration process in the radial case, i.e., between a boundary point and an interior point, is explained in section 4.3 of [10].)

A priori we expect that there are two lattice effects in the first order correction. One is that the lattice introduces a length scale (the lattice spacing). The other is that the lattice breaks rotation invariance. Given a domain  $D$  we can introduce the lattice with any orientation with respect to the domain, and the scaling limit will still be harmonic measure. However, we expect that the first order correction will depend on the orientation of the lattice with respect to the domain. We attempt to remove this dependence by averaging over rotations of the orientation of the lattice with respect to the domain. Thus the random walk models that we study are in fact rotationally invariant, and we expect that the main effect of the lattice is the introduction of a length scale. There is no doubt that this averaging over the orientation of the lattice changes the first order correction in a significant way, and so we are not actually studying the first order correction for the model in which the lattice orientation is kept fixed. Nonetheless, we see the study of this rotationally averaged model as a first step in the study of the first

order correction.

Given a domain  $D$  we fix some canonical orientation of the lattice and then rotate the lattice about the origin by an angle  $\alpha$  with respect to this canonical orientation. There is some ambiguity in how one defines the point where the random walk exits the domain. The most natural definition is to linearly interpolate between the steps of the walk so that it becomes a piece-wise linear curve in the plane, and we can then consider the first point where this curve intersects the boundary of the domain. We will refer to the distribution of this point as the discrete harmonic measure at 0 and denote it by  $\omega_{\delta,\alpha}^{M,L}(0, \cdot; D)$ . The symbol  $M$  indicates the model (ordinary random walk, random walk with no backtracking, or smart kinetic walk), and  $L$  indicates the lattice (square, hexagonal or triangular). This is a discrete measure on the finite set of sites where the bonds of the lattice with orientation  $\alpha$  and spacing  $\delta$  intersect the boundary. Now we average over the orientation by defining

$$\omega_{\delta}^{M,L}(0, \cdot; D) = \frac{1}{2\pi} \int_0^{2\pi} \omega_{\delta,\alpha}^{M,L}(0, \cdot; D) d\alpha \quad (1)$$

Because of this averaging over the orientation of the lattice,  $\omega_{\delta}^{M,L}(0, \cdot; D)$  is a continuous measure on  $\partial D$ . We let  $\omega(0, \cdot; D)$  be harmonic measure for  $D$ . We are interested in the difference  $\omega_{\delta}^{M,L}(0, \cdot; D) - \omega(0, \cdot; D)$ . We expect it to be of order  $O(\delta)$ . So we would like to compute the limit of  $[\omega_{\delta}^{M,L}(0, \cdot; D) - \omega(0, \cdot; D)]/\delta$  as  $\delta \rightarrow 0$ .

For each of the three models we carry out simulations on three lattices - square, hexagonal and triangular. Much to our surprise we find that up to an overall constant, the first order correction is the same for all three models on all three lattices. The constant depends on the model and the lattice, but not on the domain. The precise conjecture is as follows.

**Conjecture 1.** *For each model  $M$  and each lattice  $L$  there is a constant  $c_{M,L}$ , and for each simply connected domain  $D$  with smooth boundary there is a function  $\rho_D(z)$  on the boundary such that*

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} [\omega_{\delta}^{M,L}(0, \cdot; D) - \omega(0, \cdot; D)] = c_{M,L} \rho_D(z) |dz| \quad (2)$$

Here  $|dz|$  is the measure on the boundary given by Lebesgue measure with respect to arc length. The convergence is convergence in distribution. The first order difference is a signed measure, so the function  $\rho_D(z)$  is both positive and negative. Its integral along the boundary is zero.

The function  $\rho_D(z)$  is determined by the following equation. For smooth functions  $g(z)$  on the boundary,

$$\int_{\partial D} \rho_D(z) g(z) |dz| = \int_{\partial D} \frac{\partial f}{\partial n}(z) \omega(0, |dz|; D) \quad (3)$$

where  $f$  is the harmonic function in  $D$  with boundary values given by  $g$ , i.e., it is the solution of Laplace's equation with boundary data  $g$ . The derivative  $\frac{\partial f}{\partial n}$  is with respect to the inward unit normal to the domain.

Again, we should emphasize that the above conjecture is expected to be true only if we average over orientations of the lattice. If we do not average over the orientations, we expect that the first order correction depends on the orientation  $\alpha$ . We have been deliberately vague about just how smooth the boundary of  $D$  needs to be in the conjecture since we do not have any basis for a precise characterization of the smoothness needed.

Another possible definition of the random walk exit point from the domain is to take the first point of the discrete random walk outside the domain and then orthogonally project this point onto the boundary, i.e., take the point on the boundary that is closest to the first step the walk takes outside the domain. (We will only consider domains with smooth boundary, so when the lattice spacing is sufficiently small this point will be unique.) With this definition the limiting measure will still be harmonic measure. However, it is quite possible that the first order correction will be different. Surprisingly our simulations indicate that it is in fact the same.

In the next section we give a heuristic derivation of the conjecture for the ordinary random walk. The following section discusses how we compute the function  $\rho_D(z)$ . Then we discuss the results of our simulations. The paper ends with some conclusions and open questions.

## 2 Heuristic derivation of the conjecture

In this section we give a non-rigorous derivation of the conjecture for the ordinary random walk. As we will see later, this derivation only works for the square and triangular lattices. For this derivation we use yet another definition of the random walk exit point. We run the random walk until it first hits a lattice site in  $D$  which has a nearest neighbor outside of  $D$ . Then we project this site to the closest site on the boundary of the domain. The resulting site on the boundary of the domain is the definition of the random walk exit point that we will use in this section.

Let  $g$  be a function on the boundary of the simply connected domain  $D$ . Let  $f$  be the solution of the continuum Dirichlet problem

$$\begin{aligned}\Delta f(z) &= 0, & z \in D, \\ f(z) &= g(z), & z \in \partial D\end{aligned}\tag{4}$$

We assume that  $\partial D$  and  $g$  are sufficiently smooth so that  $f$  has whatever smoothness is needed below. The conjecture says that

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[ \int_{\partial D} g(z) \omega_\delta(0, |dz|; D) - \int_{\partial D} g(z) \omega(0, |dz|; D) \right] = c \int_{\partial D} \frac{\partial f}{\partial n}(z) \omega(0, |dz|; D) \tag{5}$$

where  $\omega_\delta$  is the exit distribution for one of the random walk models and  $\omega$  is harmonic measure.

Recall that  $\omega_\delta$  is the uniform average over  $\alpha$  in  $[0, 2\pi]$  of  $\omega_{\delta, \alpha}$ , where  $\omega_{\delta, \alpha}$  uses a lattice at orientation  $\alpha$ . Let  $D_{\delta, \alpha}$  be the lattice sites in  $D$  for the lattice at orientation  $\alpha$ , and let  $B_{\delta, \alpha}$  be the sites in  $D_{\delta, \alpha}$  with at least one nearest neighbor outside of  $D$ . For  $x \in B_{\delta, \alpha}$  define  $g_{\delta, \alpha}(x)$  to

be the value of  $g$  at the point on  $\partial D$  that is closest to  $x$ . (So the line from  $x$  to this point is perpendicular to the boundary.) Let  $f_{\delta,\alpha}$  solve

$$\begin{aligned}\Delta_{\delta,\alpha} f_{\delta,\alpha}(z) &= 0, & z \in D_{\delta,\alpha} \\ f_{\delta,\alpha}(z) &= g_{\delta,\alpha}(z) & z \in B_{\delta,\alpha}\end{aligned}\tag{6}$$

where  $\Delta_{\delta,\alpha}$  is the discrete Laplacian on the lattice with spacing  $\delta$  at orientation  $\alpha$ . The discrete Laplacian is given by

$$\Delta_{\delta,\alpha} h(z) = \frac{a}{\delta^2} \sum_{y:|y-z|=\delta} [h(y) - h(z)]\tag{7}$$

where  $z$  is a lattice site and the sum is over the nearest neighbors  $y$  of  $z$ . The constant  $a$  depends on the lattice and is chosen so that for any sufficiently smooth function  $h$ ,  $\Delta_{\delta,\alpha} h(z)$  converges to  $\Delta h(z)$  as  $\delta \rightarrow 0$ . (On the square lattice  $a = 1$ .) If we start a nearest neighbor random walk at 0 and run it until it hits  $B_{\delta,\alpha}$ , then the average of  $g_{\delta,\alpha}$  with respect to the resulting measure on  $B_{\delta,\alpha}$  is  $f_{\delta,\alpha}(0)$ . Similarly,  $f(0)$  is the integral of  $g$  with respect to harmonic measure on the boundary. So (5) can be rewritten as

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[ \frac{1}{2\pi} \int_0^{2\pi} f_{\delta,\alpha}(0) d\alpha - f(0) \right] = c \int_{\partial D} \frac{\partial f}{\partial n}(z) \omega(0, |dz|; D)\tag{8}$$

We now define another solution to the discrete Laplace equation with different boundary data. Let  $\hat{g}_{\delta,\alpha}$  be the function on  $B_{\delta,\alpha}$  obtained by restricting  $f$  to  $B_{\delta,\alpha}$ . Let  $\hat{f}_{\delta,\alpha}$  solve

$$\begin{aligned}\Delta_{\delta,\alpha} \hat{f}_{\delta,\alpha}(z) &= 0, & z \in D_{\delta,\alpha}, \\ \hat{f}_{\delta,\alpha}(z) &= \hat{g}_{\delta,\alpha}(z) & z \in B_{\delta,\alpha}\end{aligned}\tag{9}$$

Since  $f$  is harmonic, if it is sufficiently smooth then a simple Taylor series argument shows that  $\Delta_{\delta,\alpha} f$  is  $O(\delta^p)$  where  $p = 2$  for the square lattice,  $p = 4$  for the triangular lattice, and  $p = 1$  for the hexagonal lattice. The discrete maximum principle can then be used to show that  $f(0) - \hat{f}_{\delta,\alpha}(0) = O(\delta^p)$ . (For example, see chapter 1 of [11].) So for the square and triangular lattices which have  $p \geq 2$ , it suffices to study the difference  $f_{\delta,\alpha}(0) - \hat{f}_{\delta,\alpha}(0)$ . On the hexagonal lattice, the error in replacing  $f(0)$  by  $\hat{f}_{\delta,\alpha}(0)$  is  $O(\delta)$  which is as big as the first order correction we are trying to derive. So our derivation does not work on the hexagonal lattice. The difference  $f_{\delta,\alpha}(0) - \hat{f}_{\delta,\alpha}(0)$  solves the discrete Laplace equation with boundary data  $g_{\delta,\alpha} - \hat{g}_{\delta,\alpha}$ . For  $z \in B_{\delta,\alpha}$ , let  $H_{\delta,\alpha}(0, z)$  be the probability that the random walk (started at 0) first hits  $B_{\delta,\alpha}$  at  $z$ . Then

$$f_{\delta,\alpha}(0) - \hat{f}_{\delta,\alpha}(0) = \sum_{z \in B_{\delta,\alpha}} H_{\delta,\alpha}(0, z) [g_{\delta,\alpha}(z) - \hat{g}_{\delta,\alpha}(z)]$$

Recall that  $g_{\delta,\alpha}(z)$  is the value of  $g$  at the point on the boundary of  $D$  which is closest to  $z$ . Since  $f = g$  on the boundary, we can think of  $g_{\delta,\alpha}(z)$  as the value of  $f$  at this closest boundary

point. Recall also that  $\hat{g}_{\delta,\alpha}(z)$  is the value of  $f$  at  $z$ . So a Taylor expansion of the difference of  $f$  at these two points yields

$$g_{\delta,\alpha}(z) - \hat{g}_{\delta,\alpha}(z) = -\frac{\partial f}{\partial n}(z) \text{dist}(z, \partial D) + O(\text{dist}(z, \partial D)^2) \quad (10)$$

Generically, as we move along the boundary the relation of the points  $z \in B_{\delta,\alpha}$  to the boundary varies. For example, on the square lattice some  $z$  will have two nearest neighbors that are outside  $D$ , while other  $z$  will only have one. So the probability  $H_{\delta,\alpha}(0, z)$  that the random walk firsts hits  $B_{\delta,\alpha}$  at  $z$  will vary significantly as we move along  $B_{\delta,\alpha}$ .

We divide  $B_{\delta,\alpha}$  into segments whose length is of order  $\sqrt{\delta}$ . As the lattice spacing goes to zero, the number of lattice sites in each segment goes to infinity while the length of the segment goes to zero. So  $\frac{\partial f}{\partial n}$  is essentially constant on each segment. Let  $\frac{\partial f}{\partial n}(S)$  denote its value on segment  $S$ . So we now have

$$\begin{aligned} f_{\delta,\alpha}(0) - \hat{f}_{\delta,\alpha}(0) &\approx -\sum_S \frac{\partial f}{\partial n}(S) \sum_{z \in S} H_{\delta,\alpha}(0, z) \text{dist}(z, \partial D) \\ &= -\sum_S \frac{\partial f}{\partial n}(S) \delta(S, \alpha) \sum_{z \in S} H_{\delta,\alpha}(0, z) \end{aligned} \quad (11)$$

where the sum on  $S$  is over the segments and

$$\delta(S, \alpha) = \frac{\sum_{z \in S} H_{\delta,\alpha}(0, z) \text{dist}(z, \partial D)}{\sum_{z \in S} H_{\delta,\alpha}(0, z)} \quad (12)$$

Since the boundary is smooth and the lengths of the segments are going to zero, the portion of the boundary corresponding to a segment  $S$  is becoming essentially linear. Note that the distance from  $z$  to  $\partial D$  is of order  $\delta$ . We conjecture that as  $\delta$  goes to zero,  $\delta(S, \alpha)/\delta$  will converge to a limit that depends only on the angle of the tangent to the boundary with respect to the lattice orientation. Thus when we average over  $\alpha$ ,  $\delta(S, \alpha)$  may be replaced by some constant  $c$  times  $\delta$ . For all  $\alpha$ ,  $\sum_{z \in S} H_{\delta,\alpha}(0, z)$  may be approximated by  $\omega(0, S; D)$ . So we now have

$$\frac{1}{2\pi} \int_0^{2\pi} [f_{\delta,\alpha}(0) - \hat{f}_{\delta,\alpha}(0)] d\alpha \approx -c\delta \sum_S \frac{\partial f}{\partial n}(S) \omega(0, S; D) \quad (13)$$

After dividing by  $\delta$  this converges to

$$-c \int_{\partial D} \frac{\partial f}{\partial n}(z) \omega(0, |dz|; D) \quad (14)$$

So this non-rigorous argument suggests that with the definition of the random walk exit point we have used here, the constant in the conjecture will be negative.

We end this section with a probabilistic interpretation of our conjecture for the first order correction. Let  $A$  be a subarc of the boundary. We consider the correction for the probability

that the random walk exits through  $A$ , i.e., the probability the random walk exits through  $A$  minus the probability a Brownian motion exits through  $A$ . Up to an overall constant, our conjecture is that to first order in  $\delta$  it is given by

$$\delta \int_{\partial D} \frac{\partial f}{\partial n}(z) \omega(0, |dz|; D) \quad (15)$$

where  $f$  is the harmonic function with boundary data  $g$  which is 1 on  $A$  and 0 on  $A^c$ . (By  $A^c$  we mean the boundary points that are not in  $A$ .) The normal derivative is approximately  $(f(z + \delta n) - g(z))/\delta$ . Since the normal derivative is with respect to the inward normal,  $z + \delta n$  is a point inside  $D$  that is at a distance  $\delta$  from the boundary. Let  $\gamma^\delta$  be the curve that is a distance  $\delta$  inside  $D$  from the boundary. In other words,  $\gamma^\delta$  is the curve traced by  $z + \delta n$  as  $z$  runs over the boundary. Let  $\gamma_0^\delta$  be the part of  $\gamma^\delta$  where  $g = 0$ , and  $\gamma_1^\delta$  the part where  $g = 1$ . Then the above is approximately

$$\int_{\gamma_0^\delta} f(z) \omega(0, |dz|; D) + \int_{\gamma_1^\delta} [f(z) - 1] \omega(0, |dz|; D) \quad (16)$$

Note that  $f(z)$  is the probability that if we start a Brownian motion at  $z$ , then it exits through  $A$ . We denote this by  $P(z \rightarrow A)$ . And  $1 - f(z)$  is  $P(z \rightarrow A^c)$ . So the above is

$$\int_{\gamma_0^\delta} P(z \rightarrow A) \omega(0, |dz|; D) - \int_{\gamma_1^\delta} P(z \rightarrow A^c) \omega(0, |dz|; D) \quad (17)$$

The first term is the probability that the Brownian motion gets within  $\delta$  of the complement of  $A$  but then eventually exits through  $A$ . The second term is the probability that the Brownian motion gets within  $\delta$  of  $A$  but then eventually exits through the complement of  $A$ .

### 3 Computing the conjectured difference

In [4] we proved that for a simply connected domain  $D$  with sufficiently smooth boundary there is a continuous function  $\rho_D(z)$  on the boundary such that for sufficiently smooth  $g$ ,

$$\int_{\partial D} \frac{\partial f}{\partial n}(z) \omega(0, |dz|; D) = \int_{\partial D} g(z) \rho_D(z) |dz| \quad (18)$$

where  $f$  is the solution of Laplace's equation with boundary data  $g$ . In this section we provide some details on how to explicitly compute the function  $\rho_D(z)$  so that we can compare it with the results of our simulations.

In our simulations we always compute the cumulative distribution function (CDF) of the exit distribution rather than its density. So what we need to compute from our conjecture is the integral of  $\rho_D(z)$  over a subarc of the boundary. The five domains that we use in our simulations have the property that a ray from the origin intersects the boundary in only one

point. So we can use the polar angle  $\theta$  of a point on the boundary to parameterize the boundary. The following derivation works for any domain with smooth boundary; one just has to take  $\theta$  to be some parameterization of the boundary. Let  $g_\theta(z)$  be the boundary data that is 1 on the arc corresponding to parameter values in  $[0, \theta]$  and is 0 on the rest of the boundary. Let  $f_\theta(z)$  be the solution of Laplace's equation with this boundary data. We want to compute

$$F_D(\theta) = \int_{\partial D} \frac{\partial f_\theta}{\partial n}(z) \omega(0, |dz|; D) \quad (19)$$

Let  $h(z)$  be the unique conformal map from  $D$  to the unit disc which takes the origin to the origin and takes the point on the boundary of  $D$  with polar angle 0 to the point 1 on the boundary of the unit disc. We use  $h$  to change the above integral into an integral over the unit circle. Letting  $\phi$  be the angle variable for the unit circle,  $\omega(0, |dz|; D)$  becomes  $d\phi/(2\pi)$  and  $\frac{\partial f_\theta}{\partial n}(z)$  becomes  $\frac{\partial f_\theta^\mathbb{D}}{\partial n}(e^{i\phi})|h'(h^{-1}(e^{i\phi}))|$ . Here  $f_\theta^\mathbb{D}$  is the harmonic function in the disc with the following boundary data. Let  $z(\theta)$  be the point in  $\partial D$  with parameter  $\theta$ , and let  $\phi(\theta)$  be the polar angle of  $h(z(\theta))$ . The boundary data for  $f_\theta^\mathbb{D}(e^{i\phi})$  is 1 for  $\phi = 0$  to  $\phi = \phi(\theta)$ , and 0 elsewhere. We now have

$$F_D(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial f_\theta^\mathbb{D}}{\partial n}(e^{i\phi}) |h'(h^{-1}(e^{i\phi}))| d\phi \quad (20)$$

A straightforward computation using the Poisson kernel for the disc gives

$$\frac{\partial f_\theta^\mathbb{D}}{\partial n}(e^{i\phi}) = \frac{\sin(\phi(\theta)/2)}{2\pi \sin(\phi/2) \sin((\phi - \phi(\theta))/2)} \quad (21)$$

Thus

$$F_D(\theta) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \frac{\sin(\phi(\theta)/2)}{\sin(\phi/2) \sin((\phi - \phi(\theta))/2)} |h'(h^{-1}(e^{i\phi}))| d\phi \quad (22)$$

We must evaluate this integral numerically. Note that it is only conditionally convergent because of the singularities at  $\phi = 0$  and  $\phi = \phi(\theta)$ .

## 4 Simulations

We have performed Monte Carlo simulations to compute the exit distribution for the three different walks (the ordinary random walk, the random walk with no backtracking and the smart kinetic walk) on three different lattices (square, hexagonal and triangular) in five simply connected domains. In all cases the origin is contained in the domain. The domains are shown in figure 1 and defined as follows.

$$D_1 = \{(x, y) : \|(x, y) - (1, 0)\| < 2\},$$



$$\begin{aligned}
D_2 &= \{(x, y) : -1 < y < 2\}, \\
D_3 &= T, \\
D_4 &= \{(x, y) : \|(x, y) - (-3/4, -1)\| < 5/2, \quad y > -1\}, \\
D_5 &= \{(x, y) : (x, y) - (1/2, -1/4) \in T'\}
\end{aligned}$$

where  $T$  is the equilateral triangle with vertices  $(2, 0)$ ,  $(-1, \sqrt{3})$  and  $(-1, -\sqrt{3})$ , and  $T'$  is the equilateral triangle with vertices  $(3, 0)$ ,  $(-3/2, 3\sqrt{3}/2)$  and  $(-3/2, -3\sqrt{3}/2)$ . We always start the walk at the origin. For all three domains the domain has been scaled so that the distance from the origin to the boundary is 1.

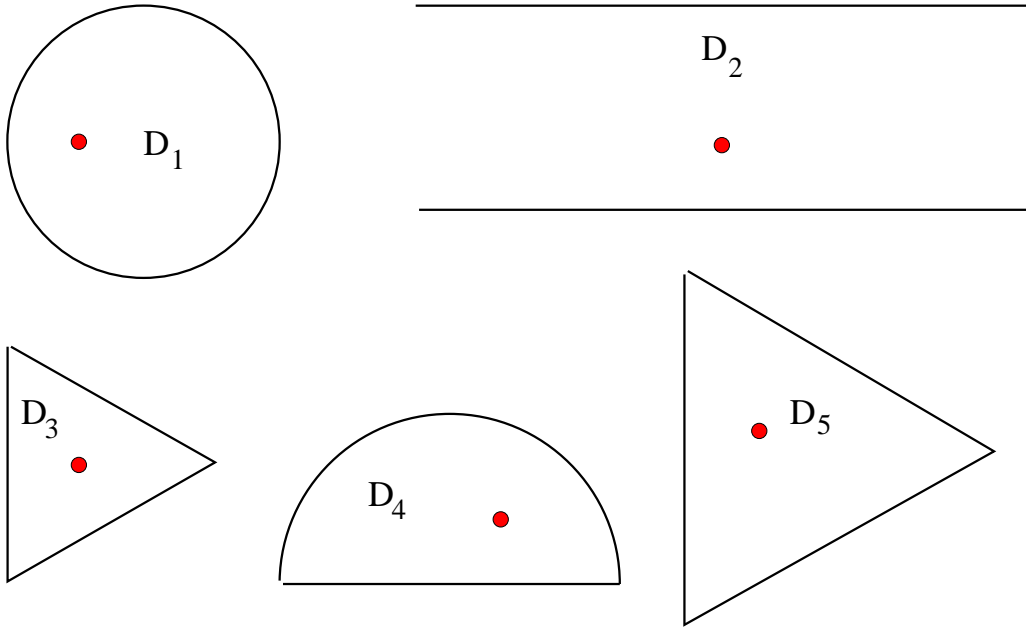


Figure 1: The five domains. The dot is the origin.

We have done simulations with lattice spacings of  $\delta = 0.04, 0.02$  and  $0.01$  for the three types of walks and the three different lattices in the five domains. For each case we generated one billion samples. The average over the orientation  $\alpha$  of the lattice is carried out as part of the Monte Carlo simulation. To generate each sample we first pick an orientation  $\alpha$  uniformly from  $[0, 2\pi]$ . Then we run the random walk model on a lattice at that orientation until it exits the domain.

In our simulations we display the results using the cumulative distribution function (CDF) rather than the density. (Computing the density from a simulation requires taking a numerical derivative and so adds further uncertainty.) The differences that we plot are the difference of the CDF for the random walk model and the CDF for harmonic measure. We plot this difference as a function of the polar angle of the boundary point with respect to the origin. We will always refer to this function as the difference function or simply the difference. Our

conjecture for this function is  $\delta c_{M,L} F_D(\theta)$  where  $F_D(\theta)$  is from the previous section. In the plots we normalize the polar angle  $\theta$  by dividing it by  $2\pi$ . For all the simulations described here we define the exit point of the random walk by linearly interpolating it between lattice sites and looking for the first time this linear interpolation crosses the boundary.

We first study how the overall magnitude of the difference depends on the lattice spacing  $\delta$ . Our conjecture says it should be proportional to  $\delta$ . In figure 2 we plot the  $L^1$  norm of the difference vs. the lattice spacing for all three models on the square and triangular lattices for domain  $D_1$ . The lines shown are fits using linear regression. The figure supports our conjecture that the magnitude is proportional to  $\delta$ . The analogous plots for the hexagonal lattice and the other domains are similar. In the plot RW stands for the ordinary random walk, RWNB stands for the random walk without backtracking, and SKW stands for the smart kinetic walk.

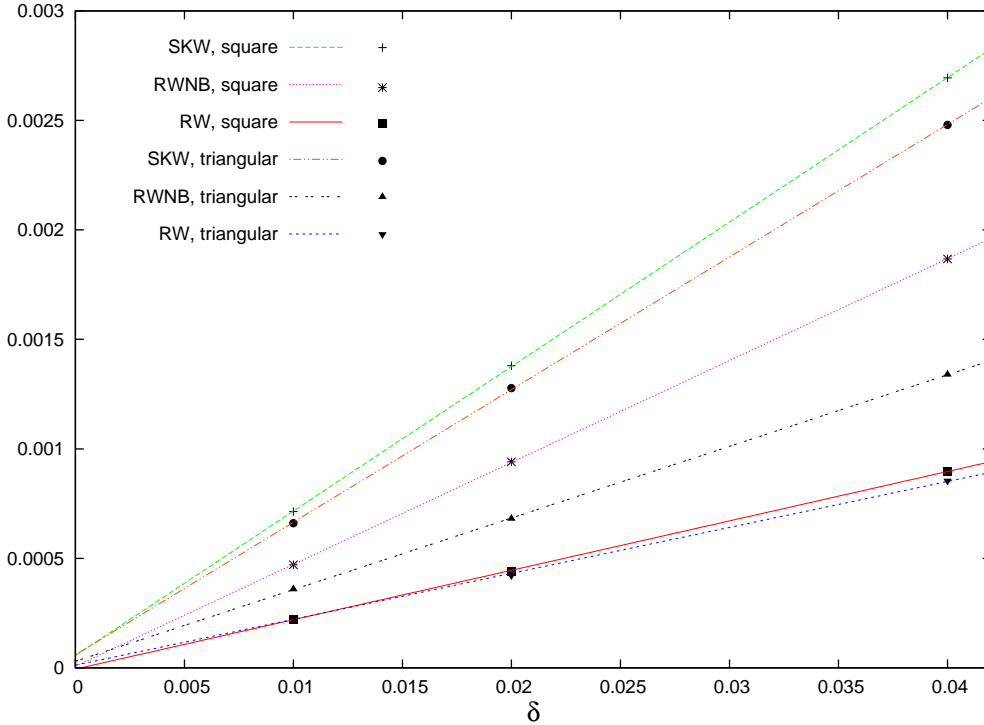


Figure 2: The  $L^1$  norm of the difference function from simulations as a function of the lattice spacing  $\delta$ . The domain is  $D_1$ .

Next we test the part of the conjecture that asserts the constant  $c_{M,L}$  only depends on the model and the lattice, not on the domain. To test this we compute the ratio of the  $L^1$  norm of the simulation difference function to the  $L^1$  norm of  $\delta\rho_D(z)$ . The conjecture says that this ratio should be independent of the domain. Table 1 shows this ratio for all three models on all three lattices for all five domains. If the conjecture is true, then the values in the table should be the same in each row. Of course, they are not exactly the same since  $\delta$  is not quite zero

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$
RW hex:	0.3996	0.3842	0.3833	0.3885	0.3641
RW sq	0.3611	0.3650	0.3704	0.3655	0.3429
RW tri	0.3668	0.3551	0.3626	0.3656	0.3539
RWNB hex	1.2108	1.1954	1.1841	1.2051	1.1458
RWNB sq	0.7625	0.7825	0.7670	0.7616	0.7376
RWNB tri	0.5794	0.5745	0.5651	0.5699	0.5500
SKW hex	1.1966	1.1710	1.1684	1.1981	1.1400
SKW sq	1.1632	1.1614	1.0951	1.1316	1.0803
SKW tri	1.0713	1.0709	1.0150	1.0431	1.0171

Table 1: Each value in the table is the ratio of the  $L^1$  norm of the simulation difference function to the  $L^1$  norm of  $\delta\rho_D(z)$ . The row labels indicate the model and the lattice, and the column labels indicate the domain.

and there are statistical errors from the simulations. The difference between the smallest and largest value in a given row ranges from less than 4% to a little more than 7%.

Finally we compare the polar angle dependence of the difference functions from our simulations with the function predicted by our conjecture. To do this we rescale all the difference functions by dividing them by  $\delta c_{M,L}$ . The conjecture says that for a given domain  $D$ , all nine of these rescaled difference functions should be equal to the function  $F_D(\theta)$ . For the values of  $c_{M,L}$  we use the average of the values in the table over the five domains. We only consider the simulations with the smallest lattice spacing ( $\delta = 0.01$ ) for this comparison. Since there are three models being simulated on three lattices, for each domain we have nine difference functions in addition to the function from the conjecture. Rather than attempt to show all ten of these functions in a single plot, we plot the three difference functions for a single model on all three lattices in a single plot. For reasons of space we only show the results for the three domains  $D_2, D_4$  and  $D_5$ . The agreement between the simulations and the conjecture for the other two domains is at least as good as the agreement for these three. The resulting nine figures are figures 3 to 11. Each figure contains four curves, but it can be difficult to distinguish them since they are so close together.

We have not shown any error bars in figures 3 to 11 to keep these figures simple. In figure 12 we show a few error bars for the simulations of the three models on the square lattice for domain  $D_2$ . The curve shown is the conjectured difference function. The error bars for the other lattices and domains are similar in size. In this figure the error bars have been rescaled by dividing them by  $\delta c_{M,L}$  just as in the previous figures. Because of this rescaling the error bars for the smart kinetic walk appear smaller than those of the random walk with no backtracking, which in turn appear smaller than those of the ordinary random walk. Before the rescaling the error bars are comparable in size. These error bars only represent the error from the Monte Carlo simulation, i.e., from the fact we can only generate a finite number of samples. There

are also errors from the fact that the lattice spacing is nonzero. They are not included in the error bars in the figure.

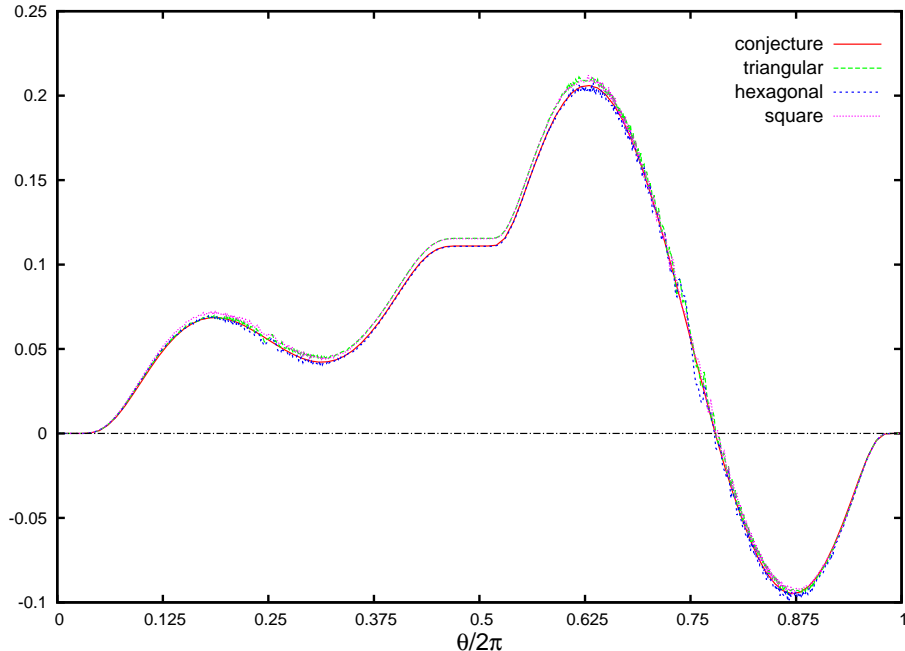


Figure 3: Rescaled difference for the smart kinetic walk for domain  $D_2$ .

## 5 Conclusions

We have studied numerically the exit distribution of the ordinary nearest neighbor random walk, the nearest neighbor random walk with no backtracking and the smart kinetic walk. We modify the usual definition of these models by also averaging over the orientation of the lattice with respect to the domain. For all three models the limiting distribution as the lattice spacing goes to zero is harmonic measure. Our simulations support the conjecture that for all three models on the three lattices studied, the difference between the random walk exit distribution and harmonic measure is, to first order in the lattice spacing  $\delta$ , given by  $\delta c_{M,L} \rho_D(z) |dz|$  where the constant  $c_{M,L}$  depends only on the lattice and the model, and the density function  $\rho_D(z)$  depends only on the domain  $D$ . Thus there is a sort of universality for this first order correction.

The most obvious open problem is to prove this conjecture; the ordinary random walk model is the natural model to consider first. In [4] the authors considered a random walk which takes place in the continuum rather than on a lattice. The steps of the walk are i.i.d. random variables which are uniformly distributed over a disc of radius  $\delta$ . Note that this model is rotationally invariant. It was proved that the conjecture holds for this model and an explicit expression for the constant which is the analog of  $c_{M,L}$  was given. This proof depended heavily

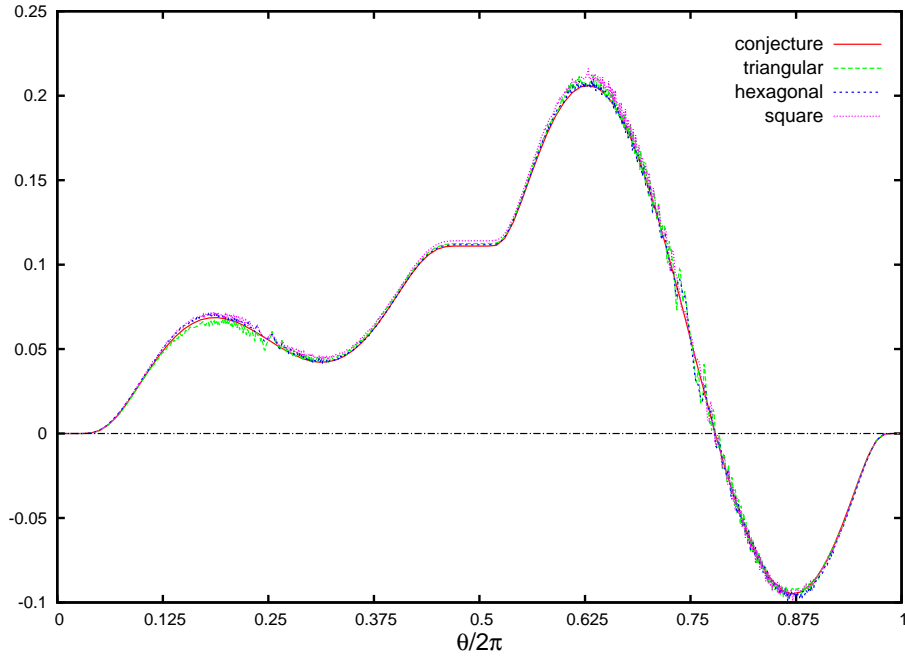


Figure 4: Rescaled difference for the random walk with no backtracking for domain  $D_2$ .

on detailed estimates of the Green's function for the model near the boundary of the domain. For the ordinary random walk the needed estimates on the Green's function appear to be much harder.

Our non-rigorous derivation of the conjecture for the ordinary random walk was based on the generator of this walk being the lattice Laplacian. The random walk with no backtracking is also Markovian if we enlarge the state space to include the direction the walk just came from. This suggests that it may be possible to give a non-rigorous derivation of the conjecture for this model using the generator. By contrast the smart kinetic walk is far from being Markovian, so a non-rigorous derivation for the conjecture for this model will require a new approach.

In principle the non-rigorous derivation of the conjecture for the ordinary random walk gives a way to compute the constant, eq. (12). It should be possible to compute this expression by Monte Carlo simulation and compare the result with the corresponding values in the table.

Perhaps the most important question for future research is to determine the first order correction when we do not average over the orientation of the lattice. If one fixes the orientation of the lattice, then the exit distribution is a discrete distribution, and this discreteness complicates the Monte Carlo computation of the first order correction in the convergence to harmonic measure. To avoid this one can average the orientation of the lattice only over a subinterval of  $[0, 2\pi]$ . Preliminary simulations show that the difference function when we average over a subinterval is significantly different from the difference function when we average over the full interval. We would like to generalize our conjecture to this case of averaging over a subinterval. In particular, it would be very interesting to determine if the first order correction still has a

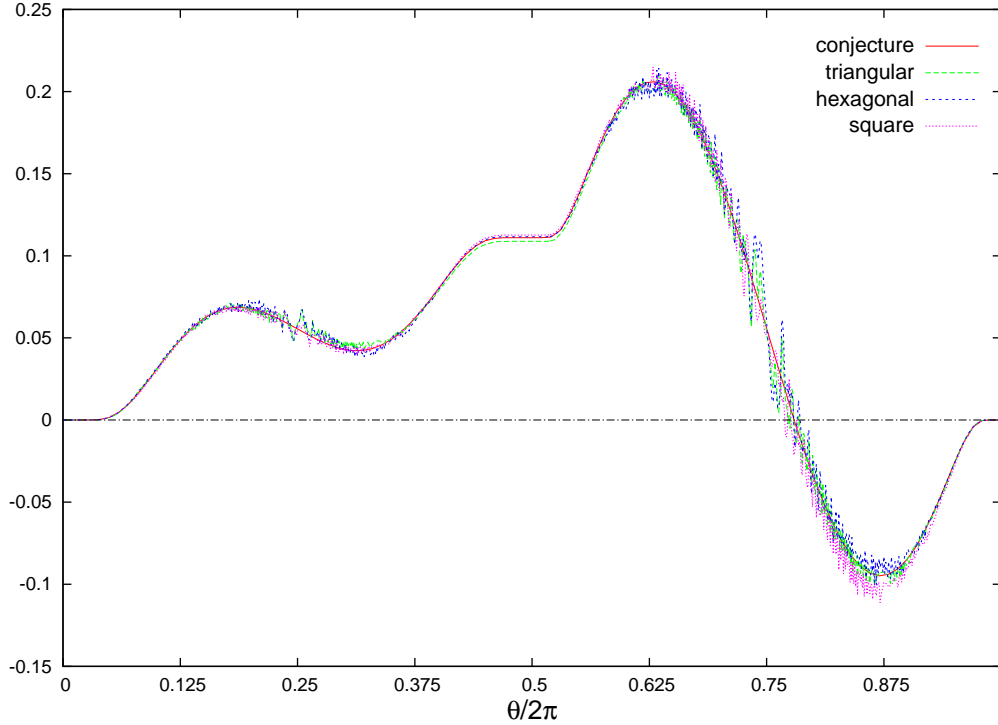


Figure 5: Rescaled difference for the ordinary random walk for domain  $D_2$ .

universal shape in the sense that the density function is the same for all three models on all three lattices except for an overall constant of proportionality.

## A Error in earlier versions

The first two versions of this paper contained an error in the simulations of the ordinary random walk on the hexagonal lattice. The error has been corrected in this version. There are two changes in the results after this correction. The most significant change is in the values in the first row of table 1. The corrected values are roughly the same size as the values in the second and third rows of the table for the random walk on the square and triangular lattices. The incorrect values for the hexagonal lattice were much larger. The other change is in the differences for the ordinary random walk on the hexagonal lattice which are plotted in figures 5, 8, 11. The overall magnitudes of these differences change significantly with the correction, but once these differences are rescaled, it is virtually impossible to distinguish the correct and incorrect plots.

The error in the simulation for the ordinary random walk on the hexagonal lattice was as follows. At each step the walk should have three equally probable choices : backtrack to the site it just came from, turn left by 60 degrees or right by 60 degrees. The error was a sign error

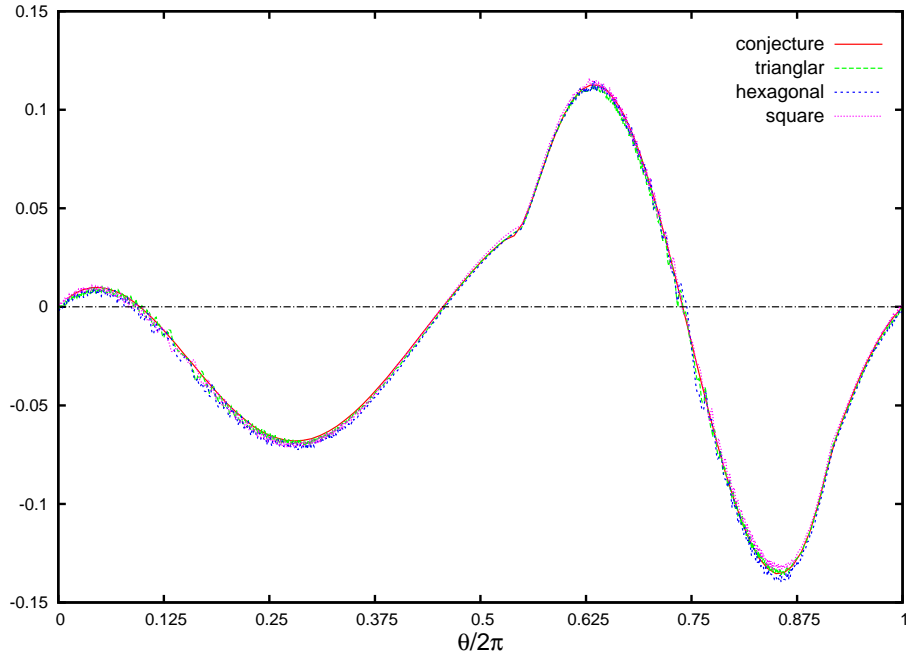


Figure 6: Rescaled difference for the smart kinetic walk for domain  $D_4$ .

so that when the walk should have backtracked it instead took another step in the direction it had just taken. It is rather surprising that even with the error the rescaled differences exhibit the same universal behavior seen in all the other models. Our understanding of this is as follows. With the error, the simulation is actually simulating a kind of random walk on the triangular lattice. At each step the walk picks with equal probability one of three possible steps: go forward, turn left by 60 degrees, turn right by 60 degrees. In the scaling limit the exit distribution of this model will be harmonic measure. The simulations indicate that our conjecture holds for this model as well, with a different constant  $c_{M,L}$  from the other models.

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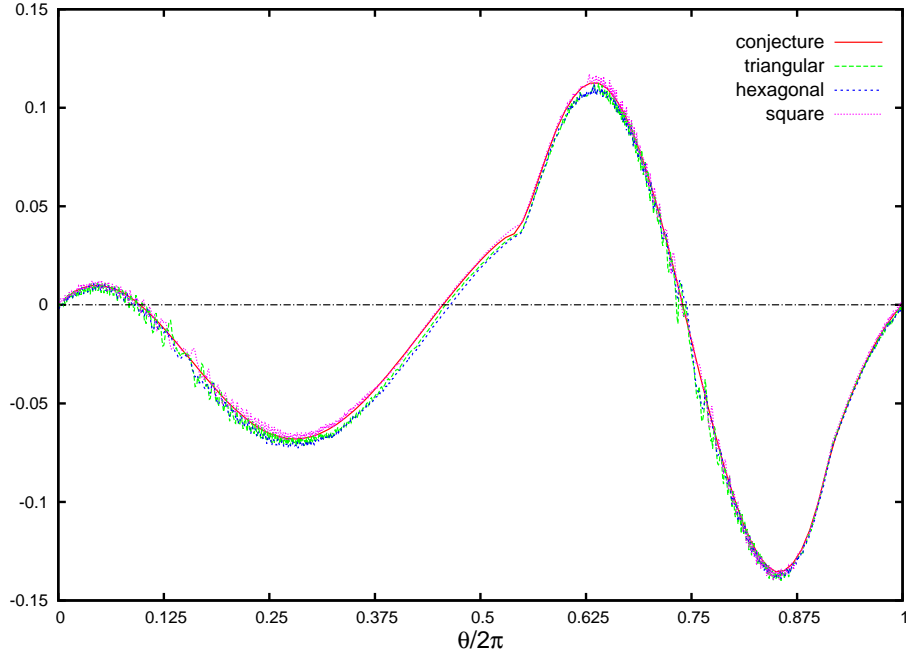


Figure 7: Rescaled difference for the random walk with no backtracking for domain  $D_4$ .

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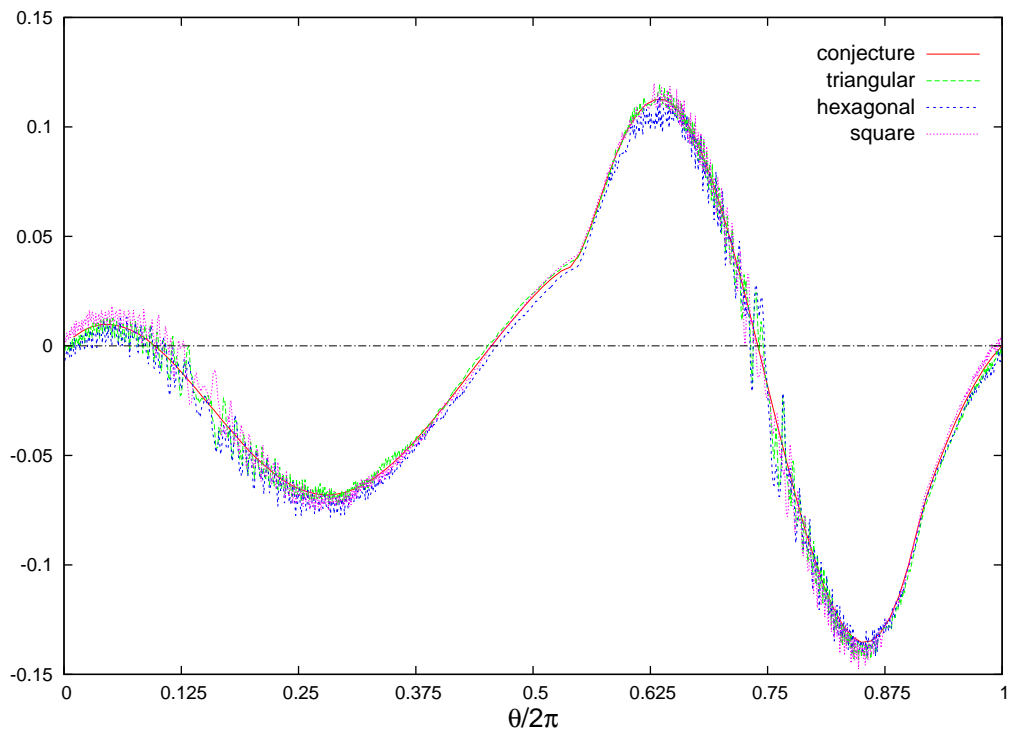


Figure 8: Rescaled difference for the ordinary random walk for domain  $D_4$ .

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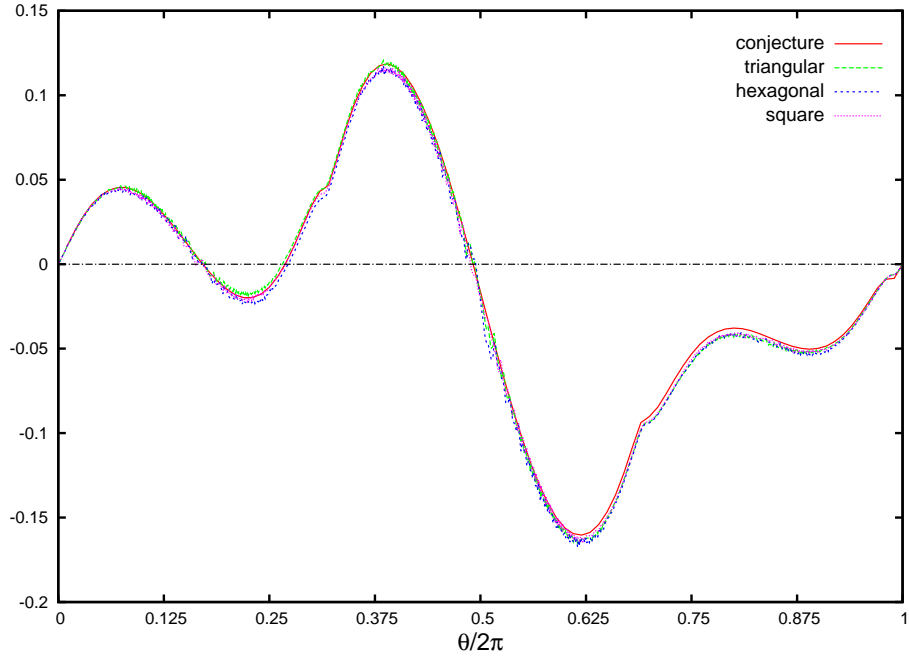


Figure 9: Rescaled difference for the smart kinetic walk for domain  $D_5$ .

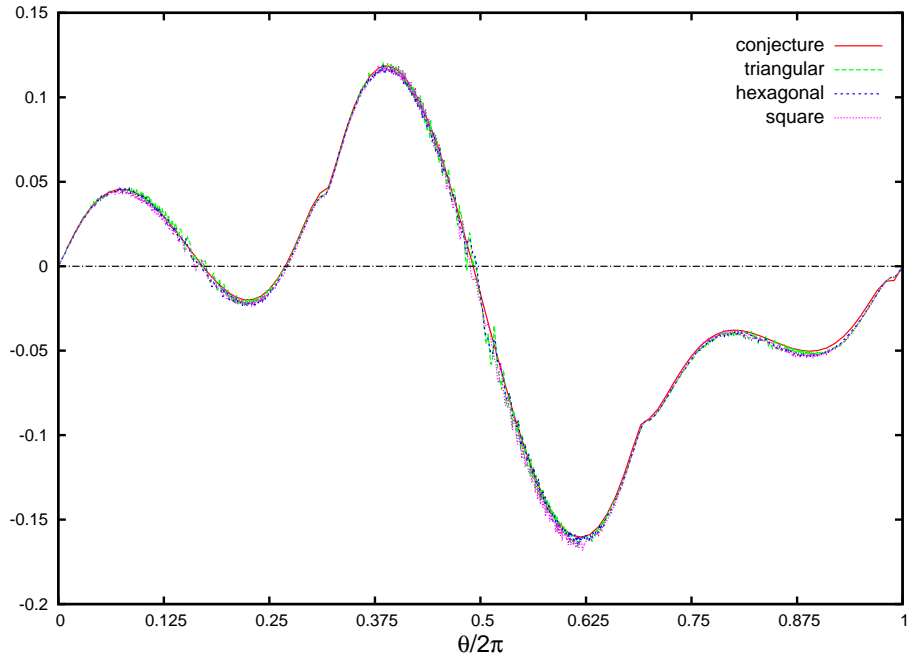


Figure 10: Rescaled difference for the random walk with no backtracking for domain  $D_5$ .

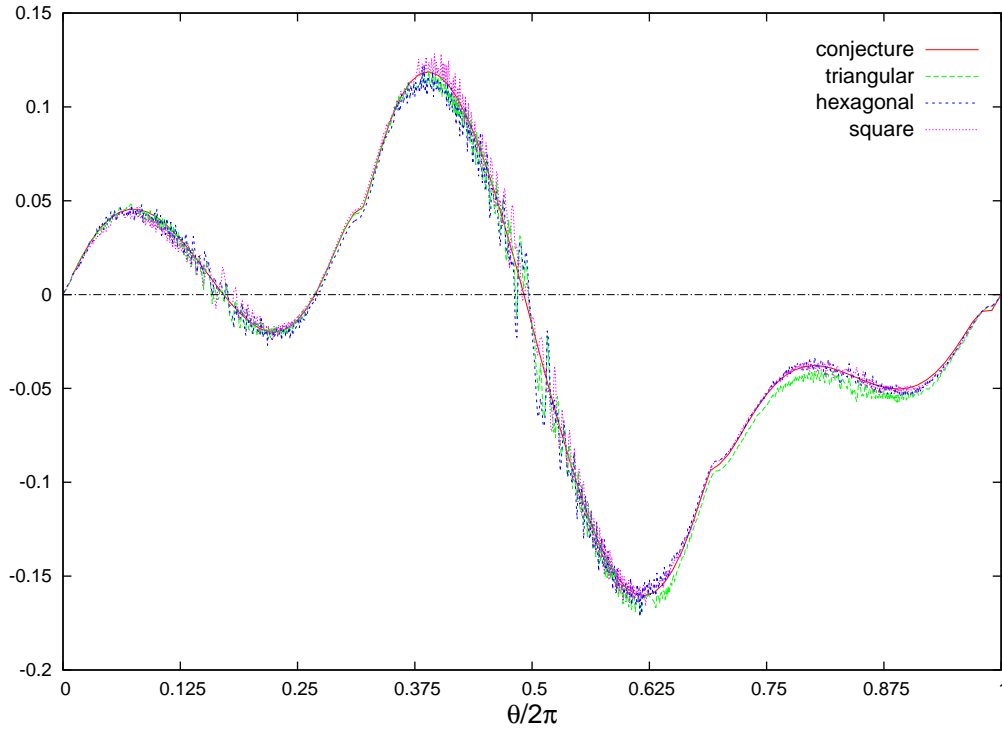


Figure 11: Rescaled difference for the ordinary random walk for domain  $D_5$ .

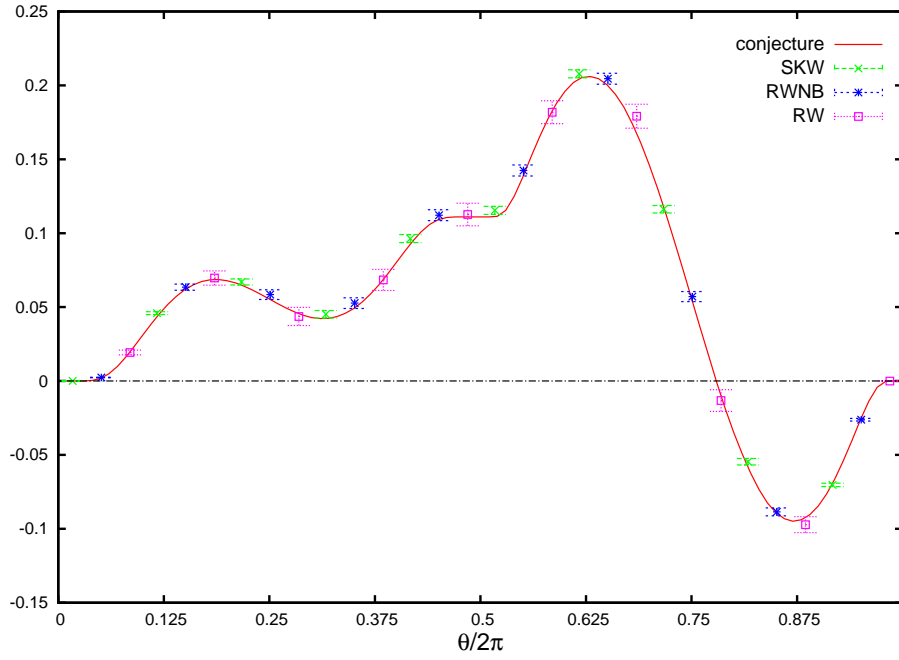


Figure 12: Error bars for the three models on the square lattice. The domain is  $D_2$ .